ON THE DISCHARGE OF A PLANE HYPERSONIC JET INTO A MEDIUM AT REST

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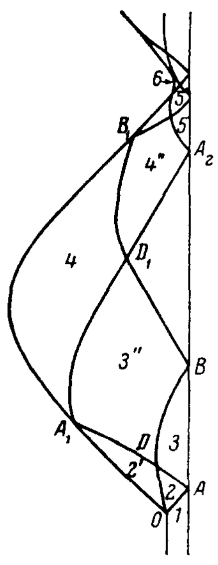
An exact solution is obtained for the problem considered here of the discharge of a jet from a straight slot using the hypersonic approximation. It is shown that for any relationship of Mach numbers in the undisturbed jet and on its boundary, shock-waves form before its first compression. The solution is obtained for an infinite sequence of discrete values of $\varkappa$.

1. Non-steady one-dimensional motion of a gas and plane irrotational hypersonic flow are defined by the same equation



$$\frac{\partial^2\chi}{\partial z^2} - \frac{4}{(\varkappa-1)^2}\frac{\partial^2\chi}{\partial\theta^2} + \frac{3-\varkappa}{\varkappa-1}\frac{1}{z}\frac{\partial\chi}{\partial z} = 0 \qquad (1)$$

where the variables have for one-dimensional non-steady motion the following meaning: z is the sound velocity and θ is the particle velocity. For plane hypersonic flow $z = 1/M = a/r$, where a is the sound velocity, r is the modulus of the particle velocity and θ is the angle between the velocity vector and the x-axis.

The variables of the flow-plane t, y or x, y are determined by the formulas

$$x \ (\text{or } t) = \frac{\varkappa-1}{2z}\frac{\partial\chi}{\partial z}, \qquad y = \theta x - \frac{\partial\chi}{\partial\theta} \qquad (2)$$

These results are based on the well-known hypersonic analogy.

For one-dimensional isentropic gas motion Formulas (1) and (2) are exact [1]; for plane steady hypersonic flow they are obtained if the quantities $h^2 z^2$ and z^2 are assumed [2] to be small compared to unity, where $h^2 = (\kappa + 1)/(\kappa - 1)$.

For values $\kappa = (2n + 3)/(2n + 1)$ where n is an integer, the general solution of Equation (1) is

$$\chi(z, \theta) = \left(\frac{\partial}{z\partial z}\right)^{n-1} \left\{\frac{1}{z}\varphi\left(z + \frac{\varkappa - 1}{2}\theta\right) + \frac{1}{z}\psi\left(z - \frac{\varkappa - 1}{2}\theta\right)\right\} \tag{3}$$

2. The solution which describes a simple wave

$$y = (\theta \pm z)x + f(\theta) \tag{4}$$

is not contained in the general solution (3) and is a singular integral of Equation (1). As a result, a flow of the form (3) may follow a region of steady flow only by going through the intermediate stage of a simple wave. On the boundary of a simple wave-characteristic, the function χ $(z, 0)$ takes the value [1]

$$\chi = -\int f(\theta)\,d\theta \tag{5}$$

3. We shall consider the problem of the discharge of a plane hypersonic jet from a straight slot into a medium at rest, which corresponds to the problem of the motion of a piston in a closed pipe when the pressure on the piston is constant. The determining parameters of the problem are: the value z_0' in the outflowing jet, the value z_1' at the free boundary of the jet, which depends on the parameters of the medium, and the half-width 1 of the jet.

The dimensionless coordinates in the plane of the flow $x = z_0'x'/l$, $y = y'/l$ depend on one parameter: $\eta = z_1'/z_0'$ and the dimensionless variables $z = z'/z_0'$ and $\theta = \theta'/z_0'$.

It is sufficient to investigate only half the jet by considering its plane of symmetry to be a solid wall.

At the exit the boundary of the jet will first be linear and emanate from the point O (see figure) with an angle $\theta = 2(\eta - 1)/(\kappa - 1)$; from O also emanates a centered rarefaction wave*. In the plane xy in region 1 there will be a uniform flow $z = 1$, $\theta = 0$, and in region 2 there will be a rarefaction wave $y = (\theta + z)x$; in region 2′ there will be a uniform flow $z = \eta$, $\theta = 2(\eta - 1)/(\kappa - 1)$. In region 3 the expansion wave interacts with the wave reflected from the wall; in this region we have [3]

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* The case of the outflow of a jet into a medium at high pressure is considered at the end of the article.

$$\chi_3(z, \theta) = \frac{(4n^2-1)(n-1)!}{2n!}\left(\frac{\partial}{z\partial z}\right)^{n-1}\left\{\frac{1}{z}\left[\left(z-\frac{\varkappa-1}{2}\theta\right)^2-1\right]^n\right\} \qquad (6)$$

or

$$\varphi_3(u) \equiv 0, \qquad \psi_3(u) = \frac{(4n^2-1)(n-1)!}{2n!}(u^2-1)^n = a(u^2-1)^n$$

Region $3'$ is a constant flow with parameters $z = 2\eta - 1$, $\theta = 0$, region $3''$ is the rarefaction wave $y = (\theta - z)x + F_3'(\theta)$; here $F_3'(\theta)$ is the value of χ_3 on the characteristic DB, i.e. for $z = 2\eta - 1 - 1/2(\kappa - 1)\theta$.

4. The flow in region 4 is determined by the boundary conditions at the characteristic A_1D_1 and at the free boundary of the jet. Regions 3 and 4 have common boundaries along the characteristics DB and A_1D_1, respectively, with one and the same simple wave. Therefore, as a result of (5) we have

$$\chi_4 = \chi_3 \quad \text{for} \quad z = 2\eta - 1 - \frac{\varkappa-1}{2}\theta \qquad (7)$$

The condition at the free boundary will be obtained from the fact that the jet boundary is a stream line and on this line (in the hypersonic approximation)

$$dy/dx = \theta \quad \text{for} \quad z = \eta$$

Since here $\partial/\partial\theta = d/d\theta$ we obtain, using (2)

$$\left(x + \theta\frac{dx}{d\theta} - \frac{\partial^2\chi}{\partial\theta^2}\right) \Big/ \frac{dx}{d\theta} = \theta \qquad \text{for} \quad z = \eta$$

which is possible only if

$$\frac{\partial^2\chi}{\partial\theta^2} - \frac{\varkappa-1}{2z}\frac{\partial\chi}{\partial z} = 0 \qquad \text{for} \quad z = \eta \qquad (8)$$

If the derivative $dx/d\theta$ becomes infinite this would mean that the jet boundary in the region of its interaction with the simple expansion wave is linear.

As a consequence of Equation (1) condition (8) may be represented also in the form

$$\left[\frac{\partial^2\chi}{\partial z^2} - \frac{1}{z}\frac{\partial\chi}{\partial z}\right]_{z=\eta} = 0 \quad \text{or} \quad \left[\frac{\partial^2\chi}{(z\partial z)^2}\right]_{z=\eta} = 0 \qquad (9)$$

5. Condition (7) defines the function $\psi_4(u)$ in the form

$$\psi_4(u) = \psi_3(u) + \Phi(b_1,\ldots, b_{n-1};\ \varphi_4(2\eta-1),\ \varphi_4'(2\eta-1),\ldots, \varphi_4^{n-1}(2\eta-1);\ u) \tag{10}$$

where Φ is a polynomial in u with proportional coefficients

$$b_1, \ldots, b_{n-1}; \qquad \varphi_4(2\eta-1),\ \ \varphi_4^1(2\eta-1),\ldots,\ \ \varphi_4^{(n-1)}\ (2\eta-1) \tag{11}$$

The constants $b_1, \ldots, b_{n-1}$ are determined by the fact that the function $\chi(z, \theta)$ is represented as the $(n-1)$th derivative of ϕ or ψ. In the expression for the function $\chi(z, \theta)$ these constants will disappear and, consequently, we may assume

$$b_1 = \ldots b_{n-1} = 0. \tag{12}$$

Condition (9) yields a linear inhomogenous equation of order $(n-1)$ with constant coefficients for the function $\phi_4(u)$. In the solution for $\phi_4(u)$ there will be $(n-1)$ arbitrary constants, which are defined by $\phi_4(2\eta-1)$, $\phi_4'(2\eta-1)$, ..., $\phi_4^{n-1}(2\eta-1)$, and by the condition of coincidence of the coordinates of the connecting point of the free jet boundary in regions 4 and 2 (one condition). This condition in general will be

$$\frac{\partial\chi_{2m}}{\partial\theta} = \frac{\partial\chi_{2m-2}}{\partial\theta} \tag{13}$$

for z and θ which correspond to the connecting point (here $2m$ is the number of the region). There are no conditions to determine the magnitudes of $\phi_4(2\eta-1)$, $\phi_4'(2\eta-1)$, ..., $\phi_4^{(n-1)}(2\eta-1)$, and therefore it may be shown by direct substitution that they are unnecessary, since they all will disappear in the expression for the function $\chi(z, \theta)$. In this manner we may assume

$$\varphi_4(2\eta-1) = \varphi_4'(2\eta-1) = \ldots = \varphi_4^{(n-1)}(2\eta-1) = 0 \tag{14}$$

From the definition of the function Φ and the relationships (12) and (14) it follows that $\Phi \equiv 0$ and

$$\psi_4(u) = \psi_3(u) = a(u^2-1)^n \tag{15}$$

According to (3) the condition (9) will be written in the form

$$\left[\left(\frac{\partial}{z\partial z}\right)^{n+1}\left\{\frac{1}{z}\varphi_4\left(z+\frac{\varkappa-1}{2}\theta\right)+\right.\right.$$

$$\left.\left.+\frac{1}{z}\psi_3\left(z-\frac{\varkappa-1}{2}\theta\right)\right\}\right]_{z=\eta} = 0 \tag{16}$$

It is easily seen that the particular solution of this equation will be

$$\varphi_4(u) = -\psi_3(u) \tag{17}$$

Indeed, according to (15), the expression

$$\frac{1}{z}\left[-\psi_3\left(z+\frac{\varkappa-1}{2}\,\theta\right)+\psi_3\left(z-\frac{\varkappa-1}{2}\,\theta\right)\right]$$

will be a polynomial in z^2 of order $(n-1)$. The general solution of (16) will be

$$\varphi_4(u) = -\psi_3(u) + a_1\exp\left(\frac{\alpha_1}{\eta}\,u\right) + a_2\exp\left(\frac{\alpha_2}{\eta}\,u\right) + \ldots + a_{n+1}\exp\left(\frac{\alpha_{n+1}}{\eta}\,u\right) \tag{18}$$

where $\alpha_1, \ldots, \alpha_{n+1}$ are the roots of the characteristic equation corresponding to (16):

$$\alpha^{n+1} - \frac{(n+1)(n+2)}{2}\,\alpha^n + \frac{(n+2)(n+1)\,n + (n+1)\,n\,(n-1) + \ldots + 3\cdot2\cdot1}{2}\,\alpha^{n-1} + \ldots$$
$$\ldots + (-\lambda)^{n+1}\frac{(2n+\lambda)!}{2^n n!} = 0 \tag{19}$$

The constants $a_1, \ldots, a_{n+1}$ are determined from conditions (13) and (14).

6. In region 5 the condition on the characteristic having a common boundary with 4,

$$\chi_5 = \chi_4 \text{ for } z = 2\eta - 1 + \frac{\varkappa-1}{2}\,\theta$$

results in the equations

$$\varphi_5(u) = \varphi_4(u) \tag{20}$$

$$\begin{aligned} &\psi_5(2\eta-1) = \psi_4(2\eta-1) = \psi_3(2\eta-1) \\ &\psi_5'(2\eta-1) = \psi_3'(2\eta-1) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\psi_5^{(n-1)}(2\eta-1) = \psi_3^{(n-1)}(2\eta-1) \end{aligned} \tag{21}$$

The condition $\partial\chi_5/\partial\theta = -1$ for $\theta = 0$ in the plane of symmetry of the flow, together with (21), determines $\psi_5(u)$ in the form

$$\psi_5(u) = \varphi_5(u) + \psi_3(u) = \varphi_4(u) + \psi_3(u) \tag{22}$$

In the subsequent regions the determination of the functions ϕ and ψ is similar to the one already investigated for the regions 4 and 5; they are all expressible through the two functions found above, namely $\psi_3(u)$

and $\phi_4(u)$. Below are given the expressions for the functions ϕ and ψ for the various flow regions.

Region	3	4	5	6	7	8	etc.
$\varphi=$	0	φ_4	φ_4	$\varphi_4+\psi_3$	$\varphi_4+\psi_3$	$\varphi_4+2\psi_3$	etc.
$\psi=$	ψ_3	ψ_3	$\varphi_4+\psi_3$	$\varphi_4+\psi_3$	$\varphi_4+2\psi_3$	$\varphi_4+2\psi_3$	etc.

7. In regions 4 to 6 the flow has the character of compression waves and, consequently, it is possible that the characteristics of one family may cross, thereby vitiating the isentropic property of the flow and causing the formation of shock-waves.

If the characteristics do not intersect in this flow, the coördinate x of the jet boundary in region 6 must be a monotonic function of θ or of the characteristic parameter $\lambda = \eta + (\kappa - 1)/2\theta$ in the interval

$$2\frac{1-\eta}{\varkappa-1} \geqslant \theta \geqslant -2\frac{1-\eta}{\varkappa-1}, \quad \text{or} \quad 2\eta-1 \leqslant \lambda \leqslant 1$$

If the characteristics do intersect, then in this region of parameter variation the function $x(\lambda)$ must have an extremum. Using the expressions given above for the functions ϕ and ψ in region 6, we obtain, according to (3) and (2)

$$x = \frac{(\varkappa-1)}{2}\left(\frac{\partial\chi_6}{z\partial z}\right)_{z=\eta} = \frac{\varkappa-1}{2}\left[\left(\frac{\partial}{z\partial z}\right)^n\left\{\frac{1}{z}\sum_{k=1}^{n+1} a_k\left\{\exp\left[\frac{\alpha_k}{\eta}\left(z+\frac{\varkappa-1}{2}\theta\right)\right] + \right.\right.\right.$$
$$\left.\left.\left. + \exp\left[\frac{\alpha_k}{\eta}\left(z-\frac{\varkappa-1}{2}\theta\right)\right]\right\}\right\}\right]_{z=\eta} = \sum_{k=1}^{n+1} b_k\left[\exp\frac{\alpha_k\lambda}{\eta} + \exp\frac{\alpha_k(2\eta-\lambda)}{\eta}\right]$$

where b_k does not depend on λ. Hence

$$\frac{dx}{d\lambda} = \sum_1^{n+1}\frac{b_k\alpha_k}{\eta}\left[\exp\frac{\alpha_k\lambda}{\eta} - \exp\frac{\alpha_k(2\eta-\lambda)}{\eta}\right]$$

It is easily seen that the root of this function is the value $\lambda = \eta$, located in the indicated interval of the variation of λ. This means that for any configuration of Mach numbers in the discharging stream and on its boundary with the medium (i.e. for any value of η) and for any κ the characteristics intersect in region 6 or earlier.

In such a manner the isentropic flow in a hypersonic stream never extends further than the first compression of the stream. Calculation shows that for $0 < \eta \leqslant 0.5$ a shock-wave is formed as a result of the intersection of the characteristics in regions 3 and 4, for $0.5 < \eta \leqslant 0.88$ in regions 4 and 4$''$, for $0.88 < \eta \leqslant 1$ in regions, 5, 5$''$ and 6.

The possibility of formation of shock-waves in the supersonic jet was explained by Pack [4] by numerical calculation using the method of finite differences for two examples of flow. The given solution is applicable directly in the case of discharge of a jet from a straight slot into a medium of high pressure. In this case, upon discharge into the medium, a shock-wave is formed, the front of which is linear. The free boundary is also linear. From the meeting point of the shock-wave, reflected from the plane of symmetry, with the free boundary emanates a central rarefaction wave and the flow considered above begins.

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